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Calculation of exit gradients at drainage ditches

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Abstract: Seepage gradients play an important role in the detachment of soil particles from the side walls of stream channels and drainage ditches. Most seepage studies have focused on water losses. Relatively few have addressed the determination of these gradients as causes of soil loss and incipient gully development. This paper presents the methodology of calculating these gradients on any point of the soil-water interface of a subsurface flow system, for which a close-form analytical solution was obtained (Römkens 2009). Such a solution was derived using conformal transformations for a situation in which a ponded surface drains by subsurface flow into a ditch with a water table lower than that of the ponded surface. The derived relationships allow a close estimate of the soil detachment forces on the wetted drainage perimeter of the stream system.

Keywords: seepage, exit gradient, subsurface flow

Introduction

There is increasing recognition that the role of subsurface flow may play a significant role in gully erosion through the increase in soil water pressures and/or seepage that adversely affects soil stability and detachment of soil particles. Recently, the results of an analytical study was published that allowed the estimation of seepage and the evaluation of pressure potentials near an incised ditch (gully) in a homogeneous aquifer of finite thickness (Römkens 2009). This article uses the results of these analyses to develop pressure exit gradient relationship at the point of water entry along the ditch surface into the stream system.

Approach

The model chosen consists of incised ditch into a flat landscape with a constant, horizontal water table higher than the water level in the ditch. The soil conducting water has a finite depth and is homogeneous and isotropic and has, therefore, a constant saturated hydraulic conductivity that is not dependent on the flow direction. The field adjacent to the ditch is

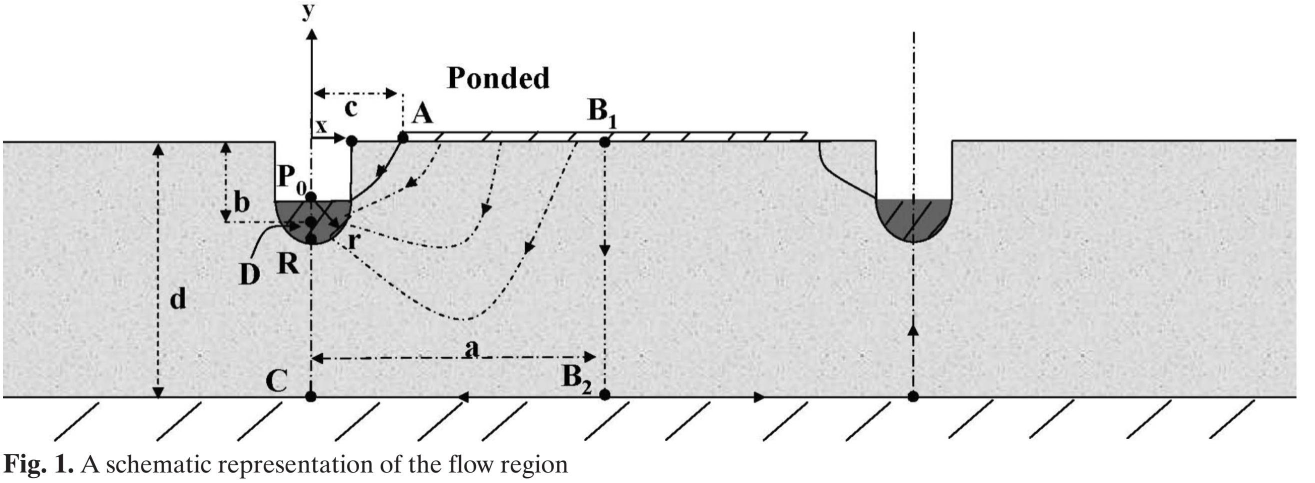

Fig. 1. A schematic representation of the flow region

ponded except for a small strip of width c along the ditch. No water is allowed to enter the soil profile through the surface of this strip. Thus, water flows through the permeable soil under a steady state regime from the field to the ditch. A physical realization of this flow region is shown in Figure 1. The incised ditch has a circularly shaped bottom which is filled with water that is maintained at a constant water level.

This flow regime is, in fact, a potential flow problem that can be described by the Laplace equation in terms of potential functions φ(x,y) and stream functions ψ(x,y). The general solution for this case has been presented by Römkens (2009) and is obtained by a series of conformal transformations:

$$\omega = -\frac{Q_1}{\pi}\ln\left[\frac{-\sqrt{\cosh\frac{c}{d}\pi-\cosh\frac{z}{d}\pi}-\sqrt{\cosh\frac{c}{d}\pi-\cosh\frac{b}{d}\pi}}{-\sqrt{\cosh\frac{c}{d}\pi--\cosh\frac{z}{d}\pi}+\sqrt{\cosh\frac{c}{d}\pi-\cosh\frac{b}{d}\pi}}\right] \quad (1)$$

where c is the width of a non-ponded strip (buffer strip) adjacent to the gully or ditch and $r = -2\cos b/d$. The latter parameter represents the relationship between the location of the drain relative to the depth of the impermeable layer. In the analysis the case for which $c = 0$ is called the drain model and the case for which c represents a finite distance is called the ditch model. Seepage calculations were made with the drain and ditch model.

Seepage gradients

The general solution shown for this flow field in Equation 2 indicates a close-form explicit expression with the aequipotential and streampotential functions on the right hand side (RHS) and the spatial coordinates on the left hand side (LHS). Given the explicit nature of the general solution, one can now calculate for each point z(x,y) the corresponding values of ω(φ,ψ). The expressions derived from Eq. 2 are:

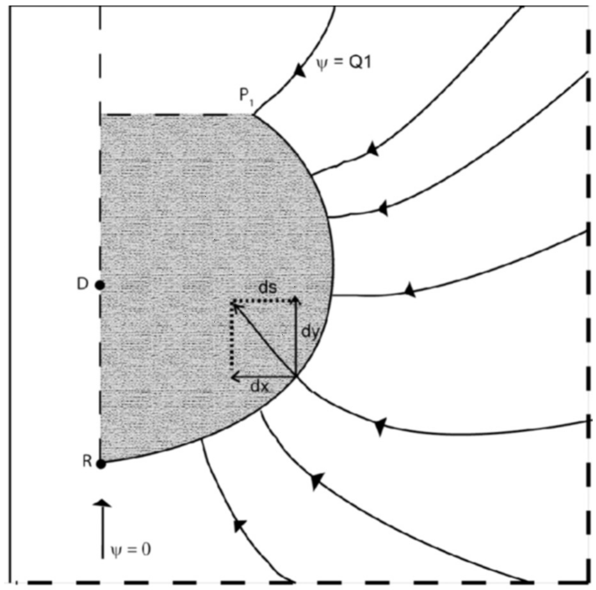

Fig. 2. Schematic representation of streamlines in the flow field near the drain

By specifying a given value for the streamline in terms of a fraction of the total seepage Q_1 in Eqs. (2) and (3), one defines in fact for each potential along the streamline the corresponding z(x,y) values. Of interest in this analysis is the seepage gradient or the potential gradient d(φ)/ds at the drain and ditch boundary, where *ds* is the spatial differential along a given streamline. Figure 2 shows a schematic representation of the exit gradient at the wetted boundary for a given streamline.

From Figure 2, the gradient along the streamline at the drain or ditch boundary is given by the expression:

$$d(\varphi)/ds=d(\varphi)/(dx + dy)=1/(dx/d(\varphi)+dy/d(\varphi)) \quad (4)$$

To calculate the gradient one needs to determine the explicit relationships x as a function of φ and y, and y as a function of φ and x, respectively. These functions can be obtained from expressions (2) and (3). To facilitate the algebraic manipulations, we re-

$$\frac{\cosh\frac{c}{d}\pi-\cos\frac{y}{d}\pi\cdot\cosh\frac{x}{d}\pi}{\cosh\frac{c}{d}\pi-\cos\frac{b}{d}\pi}=\frac{e^{\frac{-4\pi\varphi}{Q_1}}-4e^{\frac{-2\pi\varphi}{Q_1}}\cdot\sin^2\frac{\pi\psi}{Q_1}-2e^{\frac{-2\pi\varphi}{Q_1}}+1}{e^{\frac{-4\pi\varphi}{Q_1}}-4e^{\frac{-3\pi\varphi}{Q_1}}\cdot\cos\frac{\pi\psi}{Q_1}+4e^{\frac{-2\pi\varphi}{Q_1}}\cdot\cos^2\frac{\pi\psi}{Q_1}-2e^{\frac{-2\pi\varphi}{Q_1}}-4e^{\frac{-\pi\varphi}{Q_1}}\cdot\cos\frac{\pi\psi}{Q_1}+1} \quad (2)$$

$$\frac{\sin\frac{y}{d}\pi\cdot\sinh\frac{x}{d}\pi}{\cosh\frac{c}{d}\pi-\cos\frac{b}{d}\pi}=\frac{4e^{\frac{-3\pi\varphi}{Q_1}}\cdot\sin\frac{\pi\psi}{Q_1}-4e\frac{-\pi\varphi}{Q_1}\cdot\sin\frac{\pi\psi}{Q_1}}{e^{\frac{-4\pi\varphi}{Q_1}}-4e^{\frac{-3\pi\varphi}{Q_1}}\cdot\cos\frac{\pi\psi}{Q_1}+4e\frac{-2\pi\varphi}{Q_1}\cdot\cos^2\frac{\pi\psi}{Q_1}+2e^{\frac{-2\pi\varphi}{Q_1}}-4e\frac{-\pi\varphi}{Q_1}\cdot\cos\frac{\pi\psi}{Q_1}+1} \quad (3)$$

define the RHS of Eqs. (2) and (3) as $f_1(\varphi,\psi)$ and $f_2(\varphi,\psi)$, respectively. Then Eq. 2 yields the following explicit relationships for *y* and x:

$$Y = \frac{d}{\pi} \cdot \cos^{-1}\left[\frac{\cosh \frac{c}{d}\pi}{\cosh \frac{x}{d}\pi} - \frac{\cosh \frac{c}{d}\pi - \cos \frac{b}{d}\pi}{\cos \frac{x}{d}\pi} \cdot f_1(\varphi, \psi)\right] \quad (5)$$

and

$$X = \frac{d}{\pi} \cdot \cosh^{-1}\left[\frac{\cosh \frac{c}{d}\pi}{\cosh \frac{y}{d}\pi} - \frac{\left(\cosh \frac{c}{d}\pi - \cos \frac{b}{d}\pi\right)}{\cos \frac{y}{d}\pi} \cdot f_1(\varphi, \psi)\right] \quad (6)$$

Likewise, Eq. 3 yields the explicit relationships for *y* and *x*:

$$Y = \frac{d}{\pi} \cdot \sin^{-1}\left[\frac{\cosh \frac{c}{d}\pi - \cos \frac{b}{d}\pi}{\sin \frac{x}{d}\pi} \cdot f_2(\varphi, \psi)\right] \quad (7)$$

and

$$X = \frac{d}{\pi} \cdot \sinh^{-1}\left[\frac{\cosh \frac{c}{d}\pi - \cos \frac{b}{d}\pi}{\sin \frac{y}{d}\pi} \cdot f_2(\varphi, \psi)\right] \quad (8)$$

Mutual substitution of Eqs. 6 and 8 yields after several algebraic manipulations:

$$\mathrm{tg}\left(\frac{y}{d}\pi\right) = \left[-\frac{(A-B-1)}{2A} \pm \sqrt{\frac{(A-B-1)^2}{4A^2} + \frac{B}{A}}\right]^{\frac{1}{2}} \quad (9)$$

where $A = (\cosh(\pi c/d) - a.f_1(\varphi,\psi))^2$, $B = (a.f_2(\varphi,\psi))^2$, and $a = (\cosh(\pi c/d) - \cos(\pi b/d))$. Equation (10) represents an explicit relationship of *y* in terms of φ for a given ψ or streamline. The relationship dy/dφ can now readily be determined by straightforward differentiation.

Likewise, mutual substitution of Eqs. 5 and 7 yields after several algebraic manipulations:

$$\sinh\left(\frac{x}{d}\pi\right) = \left[-\frac{(A+B-1)}{2} \pm \sqrt{\frac{(1-A-B)^2}{2} + 2B}\right]^{\frac{1}{2}} \quad (10)$$

where *A* and B are defined as before. Equation (10) represents an explicit relationship of *x* in terms of φ for a given ψ or streamline. The relationship dx/dφ can now also be determined by straightforward differentiation.

Having those relationships (9) and (10), the gradient dö/ds is now determined by virtue of Eq. 4 and the location of the gradient on the wetted perimeter is determined by virtue of Eqs. (2) and (3) bearing in mind the value of the streamline ψ in terms of a fraction of Q_1 and the potential function that represents the difference between the water levels in the field and the ditch adjusted for the hydraulic conductivity. Also, the angle of the exit gradient with the positive x-axis is determined from the ratio of dφ/dy and dφ/dx. The derivation of these quantities are algebraically quite involved but are, for this case, explicit and thus are readily amenable to straightforward programming and evaluations.

In evaluating $f_1(\varphi,\psi)$ and $f_2(\varphi,\psi)$ define $u = \exp(-\pi\varphi/Q_1)$ and substitute the quantity u into the RHS of Eqs. 2 and 3. The expressions $df_1/d\varphi$ and $df_2/d\varphi$ are now readily determined from Eqs. 11 and 12 using the chain rule:

$$df_1/d\varphi = df_1/du \cdot du/d\varphi = df_1/du \cdot (-\pi\, u/Q_1) \quad (11)$$

and

$$df_2/d\varphi = df_2/du \cdot du/d\varphi = df_2/d\varphi \cdot (-\pi\, u/Q_1) \quad (12)$$

In this presentation, calculations will be made for the simple for the exit gradients at different locations of the drain and gully boundary.

References

Römkens M.J.M., 2009. Estimating seepage and hydraulic potentials near incised ditches in homogeneous, isotropic aquifer. *Earth Surfaces Processes and Landforms* 34: 1903–1914.