11 Physically Based Modelling and the Analysis of Landscape Development

D. S. L. Lawrence
Postgraduate Research Institute for Sedimentology, The University, Reading

ABSTRACT

Recent interest in the numerical simulation of drainage basin and land surface evolution raises several issues concerning the status of physically based modelling in the analysis of landscape development. Most models use a continuum description for characterizing material transport within and across system boundaries. The scale at which continuum flux models break down is often not considered in the specification of mathematical models of surface development, despite its importance in the modelling of hillslope processes contributing to channel initiation and in the field testing of physically based models. Dimensional analysis and scaling are important modelling techniques, which are still relatively underused in descriptions of landscape development, and can simplify an analysis by eliminating redundant or extraneous variables and terms in governing equations as appropriate for the scales of interest. They can also be used to identify key dimensionless quantities which must have similar values if a laboratory or other quantitative study is to be considered representative of a selected field environment. Many of the constitutive functions for material transport used in contemporary models are very similar, at least in general form, to those found in earlier studies. Current work in this area is concerned with the relative contributions of advective versus diffusive transport and with the inclusion of threshold versus non-threshold processes for channel initiation in quantitative models. The results of numerical simulations that rely on the calculation of constitutive quantities on a point-by-point basis according to a set of prescribed rules are not necessarily comparable to numerical solutions based on a discretization of the fully coupled model system. Although geomorphologists have tended to prefer the first of these approaches for its ease of implementation, it has yet to be established that this choice is a sound one and that such simulation techniques do not produce misleading or erroneous results.

The Scientific Nature of Geomorphology: Proceedings of *the 27th Binghamton Symposium in Geomorphology held 27-29 September 1996.* Edited by Bruce L. Rhoads and Colin E. Thorn.

INTRODUCTION

Physically based modelling and the mathematical formalism associated with model derivation and solution represents a fundamental research methodology used pervasively throughout the earth sciences. Its introduction in the study of landscape development is often associated with the models presented by Culling (1960, 1965), Scheidegger (1961), Ahnert (1967, 1973, 1976) and Kirkby (1971, 1976), although as pointed out by Cox (1977) credit for the first mathematical continuum model of slope development should perhaps be given to Jeffreys (1918). This chapter reviews the current status of this general methodology (which is often referred to simply as 'mathematical modelling' but which strictly speaking is more appropriately described as 'quantitative constitutive modelling') and includes discussions of the continuum hypothesis, the use of dimensional analysis and dimensionless groups to simplify models, the development of constitutive functions to characterize material transfer, and the coupling of these concepts into formal models. Rather than consider the wide array of topics within geomorphology in which this type of modelling has been applied, the emphasis herein will be on its use in describing land surface development at the hillslope to drainage basin scale. This area of work not only represents a contemporary, albeit much more quantitative, counterpart to the models of early geomorphologists such as Gilbert (1877, 1909) and Davis (1892, 1899), but is also of considerable recent interest due to a need for coupled hydrologic/geomorphic models which can be used to assess the response of surfaces to land-use or climatic changes. This latter demand motivates the development of quantitative geomorphic models which are compatible with, and contribute appropriate variables to, the somewhat more refined physical descriptions of transport processes employed by surface and subsurface hydrologists (and in some cases fluvial geomorphologists) but which also account for ongoing changes in the configuration of a surface.

The development of a mathematical model is often presented as a sequence of procedures (e.g. Thomas and Huggett 1980) in which one generates a hypothesis, constructs a mathematical model based on identified variables, simplifying assumptions and boundary conditions, develops a model solution which may be more or less quantified depending on the degree of specificity of the relationships between model variables, and finally compares model predictions with observations. If a lack of adequate correspondence between the observed and predicted variables is found, then the model is either partially or entirely redesigned to minimize the discrepancies and thus provide better predictions. Alternatively, a new set of observations which more closely reflects the variables, simplifying assumptions and boundary conditions used in the model, may be collected or compiled for the purposes of model testing. The sequence of procedures is then repeated until the objectives underlying model development are achieved. Assuming this larger framework for mathematical model development, this chapter focuses primarily on the process of model construction, although issues pertaining to solution methods or field testing are pointed out where they illustrate how physically based modelling can be more fully exploited in the study of landscape development than it currently is. The compilation and analysis of data for the field testing of hypotheses derived either from the modelling process or otherwise are not discussed herein as they are considered elsewhere in this volume.

THE CONTINUUM HYPOTHESIS

The most fundamental physical concept invoked in the development of mathematical models of landscape evolution is the principle that mass is conserved and can therefore be accounted for with a mass balance. This idea underlies not only quantitative models, but is also central to the systems approach to modelling popularized by Chorley (1962) and even to Davis' (1899), Gilbert's (1877) and Hack's (1960) discussions of land surface change with time. The 'mass' of primary interest is that portion of earth materials that extends above a reference datum and is of varying compositions, volumes and densities. One accordingly defines system boundaries which allow one to characterize the influx and outflux of mass, but which also provide for a system large enough so that the phenomenon of interest can be investigated. Given these general requirements, it is not surprising that early quantitative models of surface development focused on hillslopes (e.g. Scheidegger 1961; Ahnert 1967; Kirkby 1971) as the system boundaries can be clearly distinguished, often taking the form of a divide at the top of the slope and a stream at the bottom, with the slope laterally extending to infinity. The length of a single hillslope also seems to pose a scale that is potentially tractable in the field for model testing. It should not be overlooked, however, that these early mathematical models of slope development were not constructed in isolation from the qualitative discussions of landscape change that preceded them, and that they provided a tool for evaluating many of the concepts posed earlier in which the scale of interest had already been defined (e.g. Figure 3 of Hack 1960).

The quantification of a physical mass balance is generally based on a continuity equation which provides a foundation for the formal structure of the model. This equation usually takes a differential form, although other forms are possible, and characterizes both the influx and outflux of mass across system boundaries and the internal redistribution of material with time. This theoretical structure is thus very consistent with the scientific questions posed by geomorphologists. Problems may arise, however, in the interpretation and testing of models if the proposed continuum scale used to model the flux accounted for by the mass balance is not carefully considered in the modelling process.

The continuum hypothesis prescribes the existence of a length scale at which the properties of the system can be described by some representative average so that constitutive relations describing material flux and redistribution can be defined. Additionally, for all physical systems, an upper and lower bound may exist on the magnitude of the length scale at which a particular continuum model can be applied, and these will vary significantly between systems. A classical application of this hypothesis is in the study of fluid flow for which one distinguishes a length scale that is significantly larger than the scale at which one would 'see' the behaviour of individual molecules and which must be smaller than the scale at which large variations in physical properties would undermine the approach (Figure 11.1). Clearly, if the fluid of interest is water, then the lower bound on the appropriate length scale for defining system properties, such as fluid viscosity, will be much smaller than it is for, say, flowing granular materials. In both cases, the upper bound on the continuum length scale is often the physical boundaries of the vessel in which the fluid is contained, but it may also be more subtly associated with variations in physical properties that occur over relatively large length scales and therefore cannot readily be described by a single average value. Similarly, the continuum hypothesis underlies the description of porous media flow based on Darcy's law

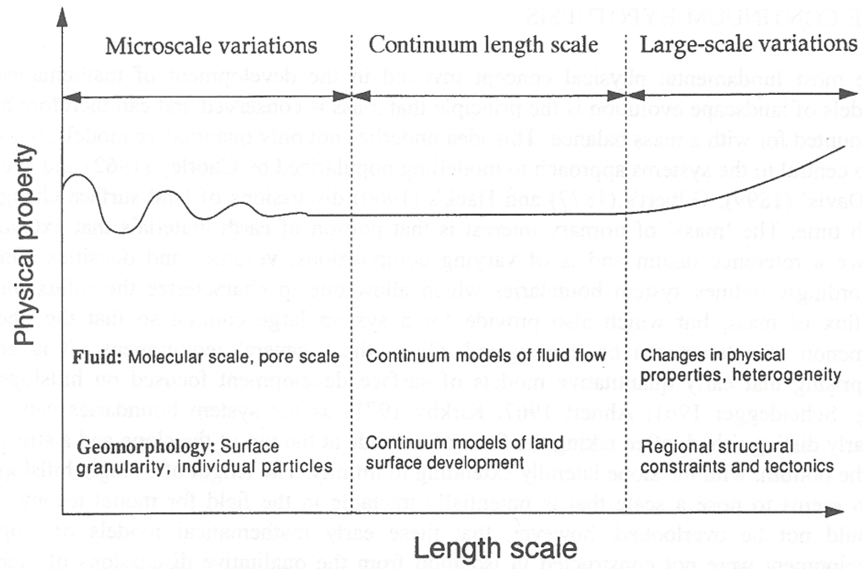

Figure 11.1 Variations in a typical physical property such as viscosity, hydraulic conductivity or surface roughness as a function of increasing length scale

such that the lower bound at which this description is applicable is given by the length scale associated with the size of individual pores and the upper bound is often influenced by regional heterogeneity in hydraulic conductivity and may be difficult to distinguish.

The application of a continuum model for land surface development presumes that one can identify a length scale at which denudational processes can be described as a material flux that is not predetermined by a superimposed structure or spatial pattern external to the model. As has long been recognized by geomorphologists, there exists a spatial scale at which regional structural features and tectonic processes significantly influence patterns of surface deformation. These features and processes effectively impose an upper bound on the length scale of applicability of many continuum models for surface evolution, as discussed by Scheidegger (1992) in somewhat different terms. The incorporation of this upper bound into quantitative denudation models is often as an initial surface configuration or as an external forcing function, but the coupling of denudation models with regional tectonic models (e.g. Koons 1989) can serve a similar role. Thus, as is the case with both the fluid and porous media examples described in the previous paragraph, upper bounds on the length scale at which a particular continuum flux model can be applied often simply represent either the presence of distinct physical boundaries or limitations on the range of variables incorporated into a particular continuum model for material flux.

The lower bound on a continuum length scale for modelling surface denudation is critical in many geomorphic studies and may be imposed by a variety of physical factors. As is true for the upper bound, the magnitude of this length scale depends upon the specificity of the model, especially the transport functions used to describe the flux, and moreover on the field environment of interest. In many cases, the granularity of the surface itself, the surface structure imparted by the presence of vegetation, or perhaps the mean path length of an individual granular particle while in transit, effectively impose a lower limit on the length scale at which a continuum transport model can be applied. This is not to say that the net or average effects of these physical factors on the overall flux of sediment cannot be incorporated into a model of landscape evolution; however, they rarely explicitly appear in constitutive model formulations for surface sediment flux. In most cases, the scale of applicability for continuum sediment transport models may be at least an order of magnitude larger than the surface detail that frequently attracts the eye in the field.

Early quantitative modellers could largely neglect the issue of identifying an intermediate length scale associated with continuum model applicability as they were primarily interested in the general relationship between process and form observed in hillslope profiles (e.g. Smith and Bretherton 1972; Kirkby 1971) and on surfaces in which a preexisting slope-valley configuration was superimposed (e.g. Ahnert 1976). However, contemporary models of simultaneous land surface and drainage basin evolution that seek to couple hillslope and channel network development must tackle this question of scale directly. The initiation and development of a drainage network itself introduces a series of multiple length scales (e.g. Tarboton et al. 1988; Rigon et al. 1993) and these may not necessarily correspond to the upper and lower bounds on a continuum length scale for modelling the denudation of unchannelized slopes. Of paramount importance is the problem of channel initiation and maintenance and specifically the length scale at which this occurs in the landscape (e.g. Dietrich et al. 1992; Montgomery and Dietrich 1992), the extent to which this length scale represents physical factors which are internal to the slope system or externally imposed (e.g. Loewenherz 1991a, 1994a; Izumi and Parker 1995), and the feasibility of incorporating a physical understanding of this scalar process into coupled models of drainage basin evolution (e.g. Willgoose et al. 1990, 1991a; Howard 1994). These questions are central to the development of physically based models for land surface development that can be applied both to further our understanding of the natural landscape and to assess the potential impacts associated with disturbances to the system. The following two sections of this chapter will consider the formulation of transport functions for use in continuum flux models that are consistent with the physical scales of interest.

DIMENSIONLESS GROUPS AND PHYSICAL SCALINGS

The development of a theoretical model, whether pursued using qualitative or quantitative methods, relies on the identification of a set of dependent variables or configurations that one seeks to explain and a set of independent variables or events that appear to contribute to that explanation. The identification of the appropriate variables for inclusion in a mathematical model and the relationships between those variables is to a certain degree an inductive process, such that the opportunity for spontaneity, serendipity and the exercise of 'common sense' is as vital to the success of this methodology as it is to more qualitative techniques (cf. Baker and Twidale 1991). Beyond the application of the conservation of mass as discussed in the previous section (and perhaps also the conservation of momentum in the case of modelling very rapid surface processes), geomorphologists are often left with relatively little guidance from physics or other basic sciences as to a quantitative form of surface weathering and transport processes. This is due largely to the differing spatial and temporal scales of interest, but also reflects the complexity of the phenomena they seek to explain. In practice, transport models are often formulated based on a combination of qualitative field or laboratory observations, the results of statistical analyses of quantified data, and physical reasoning. This section will discuss how dimensional analysis, a physical modelling technique that is relatively underused by geomorphologists, can assist in simplifying the array of variables that are incorporated into formal mathematical statements and also ensure that the resulting formal model focuses on relevant spatial and temporal scales.

The use of dimensional analysis is often thought of in its most elementary form, dimensional homogeneity, in which each group of terms in a quantitative statement must have the same dimensional representation in order for the statement to be physically meaningful. Accordingly, the physical dimensions associated with empirical coefficients in transport functions are often predetermined by the other variables used in an expression. For example, a very simple one-dimensional model for a steady-state hillslope profile in which the rate of surface transport is locally dependent on the slope is given by

$$\frac{\partial h}{\partial t}+\frac{\partial q_{\mathrm{s}}}{\partial x}=0 \qquad (1)$$

$$q_{\mathrm{s}}=kS=-k\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right) \qquad (2)$$

$$\frac{\partial h}{\partial t}=k\frac{\partial^2 h}{\partial x^2} \qquad (3)$$

where h is the elevation at a point on the surface, q_{s} is the sediment flux across the surface, S is the local surface slope, x and t are space and time coordinates respectively, and k is an empirical constant. In order for this set of equations to be dimensionally homogeneous, both the surface sediment flux, q_{s}, and the empirical constant k must have the physical dimensions $[\mathrm{L}^2/\mathrm{T}]$. This latter constant is referred to as a diffusion coefficient for surface transport due to its role in equation (3) and its associated physical dimensions.

An equally important use of dimensional analysis is in the development of dimensionless groups based on the variables contributing to a physical phenomenon. These dimensionless groups serve two major functions in physical modelling. Firstly, they provide a basis for comparing models or data independent of the underlying physical (e.g. spatial) scales. Secondly, they can be used to identify terms in a set of governing equations which may be neglected under a given set of conditions, so that the resulting formal model is specific to the physical scales of interest, and does not necessarily apply more generally to the entire range of possible values for all of the mathematical variables. According to Buckingham's $\prod$ theorem, the number of independent dimensionless groups that are necessary to describe fully a phenomenon known to involve n variables is equal to the number $n - r$, where r is usually the number of physical dimensions (such as mass, length or time) underlying the dimensional form of the variables. (For further discussion of this theorem and its application in physically based modelling see Langhaar 1951.) Thus, the total number of those groups will always be less than the number of original variables, so that this technique can be used to simplify problem statements, both those which are used in the development of formal models and those which underlie the design of field or laboratory experiments. However, the actual form that the dimensionless groups take (i.e. how the variables are arranged) is not unique, so that the choice of groupings must be guided by sound physical judgment and constant reference to the physical problem of interest.

Although process geomorphologists frequently encounter dimensionless groups developed in other scientific disciplines, such as the Reynolds number and the Froude number for describing fluid flow regimes and the Shields criterion for the incipient motion of uniform sediments, relatively little explicit attention has been given to the fundamental dimensionless groups that characterize the processes contributing to landscape evolution. They have been used for many years, however, and both 'Horton's laws' for describing drainage networks (Horton 1932, 1945) and the hypsometric integral for presenting the distribution of surface elevations within a basin (Strahler 1952) utilize dimensionless groups allowing the attributes of different systems to be compared independent of spatial scale. Similarly, the results of early quantitative models of hillslope development (e.g. Scheidegger 1961; Kirkby 1971; Ahnert 1976) are often presented in a scaled or dimensionless form. All of these examples represent geometric scalings in which variables having fundamental physical dimensions of length or area, such as the surface elevation, the distance downslope, or the basin area above a given elevation, are scaled by quantities which are constants for a given field environment and have the same fundamental physical dimension. Typical scaling factors which are used to normalize the above variables are the maximum elevation, the total slope length, or the total area of a basin, respectively. The resulting dimensionless variables usually take on values from 0 to 1, and this facilitates the comparison of systems representing very different absolute spatial scales.

As models of land surface evolution incorporate increasingly detailed and coupled physical processes, an explicit recognition of the role of dimensional analysis and scaling in model development beyond simple geometric normalizations, such as those presented above, becomes essential. Although this is beginning to emerge in recent work (e.g. Willgoose et al. 1991a; Loewenherz 1991b, 1994a), this technique has yet to be fully exploited in landscape modelling studies. Willgoose et al. (1991b) discuss the use of dimensional scaling in some detail and present a set of dimensionless groups that characterize both transient states and conditions of dynamic equilibrium for their own landscape evolution model (i.e. Willgoose et al. 1991a). Many of the fundamental physical scales underlying the derivation of those groups can be readily interpreted, such as the vertical and horizontal magnitude of the drainage contributing area; however, others, such as the runoff and channel initiation scales, require further explanation and are quite specific to the constitutive functions used in this model. Nevertheless, there are two distinct advantages in their use of a dimensionless model over one in dimensional form. First of all, the number of variables necessary for describing the system is reduced (in their case, the 30 original dimensional variables are effectively replaced by approximately 15 dimensionless groupings), and this provides for a much more parsimonious use of numerical simulations or other techniques for obtaining model results. Additionally, the dimensionless groups establish a basis for comparing model results with field or laboratory experimental studies in that they specify the parameters that must be of similar magnitude in order for two physical systems to be considered similar. This latter advantage is very aptly illustrated by the authors in their analysis of the catchment modelling laboratory experiments reported by Parker (1977) and Schumm et al. (1987).

Their analysis demonstrates that there is not necessarily a similarity of process between those experiments and typical field scale catchments due to a dominance of diffusive transport in the experimental configuration as compared with the field environment. The dimensionless group, which should be of a similar value, describes the ratio of diffusive transport to fluvial sediment transport on the surface and is approximately 100 times greater for the experimental catchment than would generally be associated with a field scale catchment.

A related use of dimensional analysis and scaling in physical modelling is in simplifying the equations governing a physical system by assigning relative magnitudes to the multiple physical scales appearing in a problem statement. This technique facilitates the mathematical analysis and interpretation of a physical problem, as demonstrated in the stability analyses for land surface development and channel initiation considered by Smith and Bretherton (1972) and Loewenherz (1991a, b, 1994a). In those models, the governing equations that are actually evaluated to assess the conditions leading to channel initiation are considerably simplified from the more general form of the governing equations. This simplification is achieved by identifying the relevant physical variables specific to the problem of interest (e.g. the hillslope length, the average depth of the surface water flux, the rate of tectonic uplift), establishing the relative magnitude of key variables (e.g. the average depth of the surface water relative to the length of the hillsope, which for most cases of land surface evolution one would assume to be very small), using the associated scalings to generate dimensionless forms of the variables, and finally distinguishing those terms in the dimensionless equations that dominate the behavior of the system from those which contribute negligibly. The associated dimensionless groups pose constraints on the range of conditions represented by the model solutions. Some of these constraints are quite simple, such as the dimensionless 'erosion time scale' specified by Smith and Bretherton (1972), which takes the form

$$t^* \sim \frac{\varepsilon \alpha t}{D_o} \tag{4}$$

where t is the time scale associated with measurable changes in the surface, α is the average effective rainfall rate on the surface, D_o scales both the vertical and horizontal lengths of the surface, ε is a dimensionless scaling factor and is posed as being very small (i.e. $\ll 1$). Thus, equation (4). directly implies that the time-dependent solutions for their model apply to surfaces for which the nominal surface lowering rate D_o/t is very small relative to the rainfall rate, α, driving the surface water flux. This constraint is clearly appropriate for many, but perhaps not all, problems of long-term surface denudation.

The physical scalings derived from a dimensional analysis can also provide a basis for expanding and coupling physical models to represent system behaviour at multiple length scales. This strategy is illustrated in the physically based model for channel initiation presented by Loewenherz (1994a) which couples a kinematic model for sediment transport at the hillslope scale with a hydrodynamic model for local sediment flux at the scale of channel initiation. The principal distinctions between this work and that of Smith and Bretherton (1972) lie in the differentiation of downslope and lateral length scales associated with surface erosional processes, and in the explicit characterization of the effects of surface water hydrodynamics on the advective transport of sediment. Two key dimensionless parameters which govern the coupling of the hydrodynamic with the kinematic model are given by

$$\delta = \frac{D}{B} = \frac{B}{L} = \left(\frac{D}{L}\right)^{1/2} \tag{5}$$

$$\Sigma = \frac{L}{UT} \tag{6}$$

where *D, L* and *B* are length scales corresponding to the depth of the surface water and the downslope and transverse (cross-slope) dimensions of the surface respectively, such that δ is very small under the physical conditions associated with channel initiation by surface flow. The timescale parameter, Σ, incorporates the mean downslope velocity of surface water, *U*, and the amount of time associated with large-scale effects on the evolution of the surface, as represented by *T*, in addition to the downslope length of the surface, *L.* This parameter is also expected to be very small for time scales at which hydrodynamic effects in the surface water flux can be expected to contribute to the local evolution of the surface. Although it is very difficult to provide useful solutions, either analytical or numerical, for the surface evolution problem if the hydrodynamic model is applied at the scale of the entire drainage surface, the use of this physical scaling facilitates the analysis by identifying the length scale at which surface water hydrodynamics will begin to contribute significantly, rather than negligibly, to the behaviour of the system.

CONSTITUTIVE FUNCTIONS FOR MATERIAL TRANSPORT

Following the identification of relevant independent and dependent variables and the clarification of the physical scales of interest, the relationship between variables must be specified if one seeks to generate model solutions. In developing models of denudational processes, one is primarily concerned with identifying constitutive functions for weathering and surface transport processes. Although the earliest mathematical models of slope profile development neglect physical discussions of the constitutive functions used in solving those models, both Ahnert (1967, 1976, 1977) and Kirkby (1971, 1976) pay careful attention to this issue in the development of their models and incorporate the findings of contemporaneous research in the specification of transport functions. The constitutive functions for sediment transport that are currently used to simulate landscape evolution (e.g. Willgoose 1991; Howard 1994) are very similar in their general form to those Ahnert and Kirkby originally proposed. There has nevertheless been considerable recent attention (Dietrich and Dunne 1993; Kirkby 1994) given to the different classes of transport functions that need to be quantified in order for a model of landscape evolution to be representative of the range of conditions occurring in the natural environment. This section will highlight some of the issues arising out of those discussions, particularly the differentiation of advective and diffusive transport processes, the use of general versus specific functional forms in analysis, and threshold versus non-threshold processes and their relationship to the more general issues of the continuum length scale underlying the physically based model.

Many of the constitutive models for material transport which are used in modelling land surface development can be reduced to the general set of variables given by

$$q_s = f(S, q_w) \tag{7}$$

where q_s is the sediment flux, S is the local surface slope and q_w is the surface water flux. This basic functional relationship is often further simplified by using the contributing area per unit contour width as a surrogate for the amount of surface water that will accumulate at a point, so that material transport is given entirely as a function of surface geometry. The variables presented in equation (7) are often used in the somewhat more specific form given by

$$q_s = F(S^n, q_w^m) \tag{8}$$

where n and m are empirical constants usually assumed to be > 0, and F is a linear or bilinear function of the variables S_n and q_w^m. As discussed by Kirkby (1971) and Smith and Bretherton (1972), this form of the equation for sediment flux reflects reported field and laboratory observations of a range of sediment transporting processes.

Two very different system behaviours can be distinguished based on the relative sensitivity of the sediment transport to the surface slope versus the surface water flux, as measured by the magnitude of the partial derivatives of the transport function F, with respect to S versus q_w (Smith and Bretherton 1972; Kirkby 1980). The first class of physical processes is generally diffusive in character in that material transport occurs in response to a gradient, which in this case is given by the slope of the surface elevation. Diffusive processes tend to stabilize physical systems as they eliminate strong gradients by locally redistributing material from zones of high concentration to those of lower concentration, thus reducing the magnitude of the gradient. In models of land surface development, diffusive processes contribute to the general smoothing of the surface in that they preferentially erode the steepest elements of the landscape. The second class of processes, representing those transport phenomena which rely on the aid of a transporting medium, such as the surface water, q_w, to move material through the system, are advective or concentrative in character. In contrast to diffusive processes, these processes contribute to the local incision and dissection of the surface, both by providing a mechanism for the removal of material from the system which may be independent of the local surface slope and by enhancing the tendency for local surface water accumulation which then increases the capacity for material entrainment and removal. This differentiation between diffusive and advective sediment transport is broadly analogous with functional forms of transport models used in other areas of physical science, such as heat transfer and porous media flow. It is of both physical and mathematical significance as the coupling of a strictly diffusive transport function with a continuity equation results in a parabolic partial differential equation, while a purely advective system is hyperbolic in character. This distinction in turn has implications for the solution techniques, either analytical or numerical, which are most appropriate for evaluating system behaviour. (For more thorough discussions of this topic, see e.g. Pinder and Gray 1977 or Ames 1992 for the case of numerical techniques; Hassani 1991 for the case of analytical methods.)

For two-dimensional models of surface development, the material transport function must specify both the magnitude and local direction of the sediment flux and is often of the form given by

$$\mathbf{q}_{\mathrm{s}} = F\left(S^{m}, q_{\mathrm{w}}^{m}\right)\mathbf{m} \tag{9}$$

where **m** is the direction of the local slope and thus determines the pathways for sediment across the surface. However, if the local pathways associated with advective sediment flux are to be explicitly distinguished from those associated with the diffusive flux, as is necessary for evaluating channel initiation (Loewenherz 1991b, 1994a), then the constitutive function must include this distinction and thus be written as

$$\mathbf{q}_{\mathrm{s}} = F_{A}\left(S^{n_1}, q_{\mathrm{w}}^{m_1}\right)\mathbf{r} + F_{D}\left(S^{n_2}, q_{\mathrm{w}}^{m_2}\right)\mathbf{m} \tag{10}$$

where **r** is the local direction of the surface water, F_A is the magnitude of the advective component of sediment transport, F_D is the magnitude of the diffusive component, and the subscripts 1 and 2 are used to distinguish the empirical constants associated with advective transport and diffusive transport respectively. It is important to note that equations (9) and (10) are still rather general constitutive functions as they do not indicate specific values for n and m, nor do they specify multiplicative or additive constants for the variables. However, they can be used in conjunction with constraints on the behaviour of the derivatives of the functions to provide some very useful qualitative insights into the processes of landscapes development and especially into the relative roles of diffusive versus advective transport in channel initiation (e.g. Smith and Bretherton 1972; Kirkby 1980; Loewenherz 1991a, 1994a). Furthermore, and of particular relevance in light of the 'detachment-limited' simulation model recently presented by Howard (1994), the general formulations given by equations (9) and (10) neither neglect nor preclude the inclusion of constant factors or coefficients into a fully quantified problem statement. They are thus not strictly limited to the analysis of transport-limited conditions, although much previous work has relied on this assumption. Accordingly, the addition of a critical shear stress as a specified constant in the sediment transport function does not in itself establish a new class of solutions that are distinct from those encompassed by the analysis of equation (9). The magnitude of the physical scales implied by these additional constants (i.e. just how large the transporting capacity must be before sediment transport will occur, the time scale over which one can anticipate that significant transport will occur, and the size of the surface area likely to be affected once transport is initiated) may, however, have implications for the continuum flux model used in the analysis.

Recent discussions of hillslope transport processes, especially those contributing to channel initiation, point towards a fundamental distinction between threshold and non-threshold processes in landscape development (Dietrich and Dunne 1993; Kirkby 1994; Montgomery and Dietrich 1989). Processes characterized by distinct thresholds contribute to the initiation of surface channels with well-defined banks, and non-threshold, more 'gradual', processes are associated with valley formation. In other types of physical problems, thresholds often represent a discontinuity in the equations or conditions governing transport, such as those which are associated with a shock wave in a compressible flow. An incipient surface channel with a well-defined channel head and banks often represents a morphological discontinuity (e.g. see illustrations in Montgomery and Dietrich 1989; Dietrich and Dunne 1993), and the equations governing the flux of water and sediment across the adjacent unchannelized surface will be different from those associated with transport in the channel. Therefore, the concept of a morphological threshold may be quite appropriate in these circumstances, particularly at this scale of resolution. However, for the purposes of modelling, the more fundamental question is to what extent can certain types of hillslope processes that ultimately result in the initiation of surface channels (e.g. surface wash and landsliding) be differentiated as threshold versus non-threshold processes. The resolution of this issue seems to be directly linked to the scale of observation of sediment-transporting events as compared with the continuum length scale that underlies the physically based model. In other words, although sheetwash erosion is characterized as a continuum process in most models of land surface development, the microscale process consists of a series of discrete, 'threshold-based' events. Similarly, in environments where landsliding is the dominant hillslope-forming process, land surface morphology is often the product of numerous discrete events, such that in principle, there may be a larger length scale at which this ongoing series of events could also be modelled as a continuum process. The distinction between these two categories is thus directly related to the magnitude of the length scale of interest relative to the typical length scale that characterizes surface deformation during a single event. If the magnitude of these scales is similar, then an appropriate continuum scale cannot be defined. However, if the length scale that is being modelled is much larger than that which is affected by microscale processes (which may or may not include local thresholds), then the continuum hypothesis is valid and the aggregate effects of this process on surface development can be defined using a continuum model.

MATHEMATICAL MODELS OF LAND SURFACE EVOLUTION

Formal mathematical models of land surface development are usually derived by applying the continuity equation to account for the transfer of mass within the system in the general form

$$\frac{\partial h}{\partial t} = -(\nabla \cdot q_s)\mathbf{m} + T \qquad (11)$$

where h is the surface elevation, q_s *is* the sediment flux that is given by a constitutive function similar to those discussed in the previous section, **m** is a vector describing the local surface gradient, and T represents any external forcing, such as tectonic uplift, which is contributing to time-dependent changes in the configuration of the land surface. The divergence operator, $(\nabla\cdot)$, constrains the redistribution of mass, and takes a variety of specific forms, depending on the spatial coordinate system which is used in the problem. An additional continuity equation is often also employed to account for the flux of water across the surface, so that it can be described as a local variable for use in the constitutive function. The dependent variable, *h,* is the actual elevation of the surface at a point in space and time so that this equation describes the relationship between process (the sediment flux) and form (the elevation) for evolving surfaces. When the constitutive function consists only of diffusive (i.e. slope-dependent) processes, equation (11) reduces to a diffusion equation, and the time-dependent behaviour of the landscape is analogous to other diffusive physical processes, such as the molecular diffusion of solutes. The conditions of dynamic equilibrium are described by simply setting the time rate of change of the surface, i.e.$\partial h/\partial t$, equal to zero, so that the system represents a balance between tectonic uplift, T, and the flux of sediment through the system.

Beyond the simple case of a strictly diffusive system in dynamic equilibrium, it can be surprisingly difficult to provide actual mathematical solutions, either analytical or numerical, for the time-dependent surface evolution problem described by equation (11) for cases which are of interest to geomorphologists. This is due to several factors, including:

1. The multidimensional nature of the problem;
2. The relatively nonlinear character of the transport functions for the surface sediment flux which arises from the exponents on S and q_w in equations (9) and (10);
3. The presence of advective transport which precludes the modelling of the system as strictly diffusive and changes the governing equation from one that is parabolic to one that may exhibit both parabolic and hyperbolic behaviours;
4. The spatially non-uniform behaviour of transport processes, particularly as associated with the general downslope increase in the importance of advective transport as surface water progressively accumulates from upslope contributions;
5. The physical discontinuity which may be associated with the presence of surface channels.

Full mathematical solutions for the governing equations, which consider multidimensional surfaces, encompass both diffusive and advective transport, and incorporate the tendency for surface channel initiation and development, are thus virtually non-existent in the geomorphic literature.

Two alternative approaches are, however, used to effectively 'solve' and thereby obtain results for landscape evolution models. The first of these is represented by the general technique of numerical simulation, in which a set of rules which describe the processes of surface erosion on a local basis and are derived from the constitutive functions and a mass balance accounting (e.g. Ahnert 1976; Kirkby 1986; Howard 1994) is applied to a lattice of points representing the surface and then reapplied at subsequent time steps, so that the problem is marched forward in time. The simulation technique is very attractive due to its intuitive character, its low demand on computational capacity (as the large matrices used in the simulation are never inverted as they would be under many numerical solution schemes), and the ease with which subsystems can be coupled to generate a larger model system. However, although rule-based simulation results are often considered by geomorphologists to be analogous to an analytical or at least an approximate numerical solution for the fully coupled differential equations governing the system, we have yet to establish the range of conditions for which this is in fact true. Of particular concern is the general nonlinear character of the transport functions and the associated potential for instability in the physical system, which in many other areas of physically based modelling (e.g. fluid dynamics) virtually precludes the use of simple simulation techniques. As models for surface evolution become increasingly sophisticated, and particularly as mechanisms for channel initiation are identified and incorporated into constitutive functions, the reliability of simulation results relative to those provided by full numerical solutions based on a discretization of the fully coupled model system deserves attention.

Analytical techniques, such as asymptotic and stability analyses, which evaluate various aspects of the system behaviour within well-defined limits (e.g. Smith and Bretherton 1972; Loewenherz 1991a, b; 1992, 1994a, b) have also been used to obtain results from land surface evolution models. These methods, while often unable to provide complete solutions, can nevertheless establish important constraints on the behaviour of the physical system (e.g. conditions associated with tendency for surface channel initiation) and can also be used to check the validity of numerical results. Although they remain relatively inaccessible to many geomorphologists, their role may become more prominent as other cognate disciplines (e.g. geophysics, civil engineering) continue to take an increasing interest in landscape evolution and as the availability and use of reliable computer software capable of algebraic and other symbolic manipulation (e.g. Mathematica by Wolfram Research, Inc.) become more widespread.

REFERENCES

Ahnert, F. 1967. The role of the equilibrium concept in the interpretation of landforms of fluvial erosion and deposition, in *L'evolution des versants,* edited by P. Macar, *Les Cong, et Colloques de l'Univ. de Liege,* **40***,* 23-41.

Ahnert, F. 1973. COSLOP 2 - a comprehensive model program for simulating slope profile development, *Geocom Bulletin,* **6***,* 99-122.

Ahnert, F. 1976. Brief description of a comprehensive three-dimensional process-response model of landform development, *Zeitschrift für Geomorphologie,* NF Supplementband, **25***,* 29-49.

Ahnert, F. 1977. Some comments on the quantitative formulation of geomorphological processes in a theoretical model, *Earth Surface Processes,* **2***,* 191-201.

Ames, W.E 1992. *Numerical Methods for Partial Differential Equations,* 3rd edn, Academic Press, London, 451 pp.

Baker, V.R. and Twidale, C.R. 1991. The reenchantment of geomorphology, *Geomorphology,* **4***,* 73-100.

Chorley, R.J. 1962. Geomorphology and general systems theory, *US Geological Survey Professional Paper 500-B.*

Cox, N.J. 1977. A note on a neglected early model of hillslope development, *Zeitschrift für Geomorphologie,* NF Supplementband, **21***,* 354-356.

Culling, W.E.H. 1960. Analytical theory of erosion, *Journal of Geology,* **68***,* 336-344.

Culling, W.E.H. 1965. Theory of erosion on soil-covered slopes, *Journal of Geology,* **71***,* 127-161.

Davis, W.M. 1892. The convex profile of bad-land divides, *Science,* **20***,* 245.

Davis, W.M. 1899. The geographical cycle, *Geographical Journal,* **14***,* 481-504.

Dietrich, W.E. and Dunne, T. 1993. The channel head, in *Channel Network Hydrology,* edited by K. Beven and M.J. Kirkby, Wiley, Chichester, pp. 175-219.

Dietrich, W.E., Wilson, C.J., Montgomery, D.R., McKean, J. and Bauer, R. 1992. Erosion thresholds and land surface morphology, *Geology,* **20***,* 675-679.

Gilbert, G.K. 1877. *Report on the Geology of the Henry Mountains,* US Geological Survey, Washington, DC 160 pp.

Gilbert, G.K. 1909. The convexity of hilltops, *Journal of Geology,* **17***,* 344-350.

Hack, J.T. 1960. Interpretation of erosional topography in humid temperate regions, *American Journal of Science,* **258A,** 80-97.

Hassani, S. 1991. *Foundations of Mathematical Physics,* Allyn and Bacon, Boston, 918 pp.

Horton, R.E. 1932. Drainage basin characteristics, *Transactions of the American Geophysical Union,* **13**, 350-361.

Horton, R.E. 1945. Erosional development of streams and their drainage basins; hydrophysical approach to quantitative morphology, *Bulletin of the Geological Society of America,* **56***,* 275-370.

Howard, A.D. 1994. A detachment-limited model of drainage basin evolution, *Water Resources Research,* **30**, 2261-2285.
Izumi, N. and Parker, G. 1995. Inception of channelization and drainage-basin formation-upstream-driven theory, *Journal of Fluid Mechanics,* **283**, 341-363.
Jeffreys, H. 1918. Problems of denudation, *Philosophical Magazine,* **36**, 179-190.
Kirkby, M.J. 1971. Hillslope process-response models based on the continuity equation, in *Slopes: Form and Process,* edited by D. Brunsden, Institute of British Geographers, Special Publication, 3, 15-30.
Kirkby, M.J. 1976. Deterministic continuous slope models, *Zeitschrift für Geomorphologie,* NF Supplementband, **25**, 1-19.
Kirkby, M.J. 1980. The stream head as a significant geomorphic threshold, in *Thresholds in Geomorphology,* edited by D.R. Coates and J.D. Vitek, Allen and Unwin, London, pp. 53-73.
Kirkby, M.J. 1986. A two-dimensional simulation model for slope and stream evolution, in *Hillslope Processes,* edited by A.D. Abrahams, Allen and Unwin, London, pp. 203-222.
Kirkby, M.J. 1994. Thresholds and instability in stream head hollows: a model of magnitude and frequency for wash processes, in *Process Models and Theoretical Geomorphology,* edited by M.J. Kirkby, Wiley, Chichester, pp. 294-314.
Koons, P.O. 1989. The topographic evolution of collisional mountain belts: a numerical look at the Southern Alps, New Zealand, *American Journal of Science,* **289**, 1041-1069.
Langha, H.L. 195 1. *Dimensional Analysis and Theory of Models,* Wiley, New York, 166 pp.
Loewenherz, D.S. 1991a. Stability and the initiation of channelized surface drainage: a reassessment of the short wavelength limit, *Journal of Geophysical Research* **96(B**), 8453-8474.
Loewenherz, D.S. 1991b. Stability and the role of length scale dependent material transport in the initiation and development of surface drainage networks, Ph.D. thesis, University of California, Berkeley, 206 pp.
Loewenherz, D.S. 1992. Finite amplitude development of surface incisions derived from sheetwash erosion, *EOS, Transactions of the American Geophysical Union,* **73**, Fall Meeting Supplement, 212.
Loewenherz, D.S. 1994a. Hydrodynamic description for advective sediment transport and rill initiation, *Water Resources Research,* **30**, 3203-3212.
Loewenherz, D.S. 1994b. Theoretical constraints on the development of surface rills: mode shapes, amplitude limitations and implications for non-linear evolution, in *Process Models and Theoretical Geomorphology,* edited by M.J. Kirkby, Wiley, Chichester, pp. 315-333.
Montgomery, D.R. and Dietrich, W.E. 1989. Source areas, drainage density and channel initiation, *Water Resources Research,* **25**, 1907-1918.
Montgomery, D.R. and Dietrich, W.E. 1992. Channel initiation and the problem of landscape scale, *Science,* **255**, 828-830.
Parker, R.S., 1977. Experimental study of drainage basin evolution and its hydrologic implications, Ph.D. thesis, Colorado State University, Fort Collins.
Pinder, G.F. and Gray, G. 1977. *Finite Element Simulation in Surface and Subsurface Hydrology,* Academic Press, New York, 295 pp.
Rigon, R., Rinaldo, A., Rodriguez-Iturbe, I., Bras, R.L. and Ijjasz-Vasques, E. 1993. Optimal channel networks-a framework for the study of river basin morphology, *Water Resources Research,* **29**, 1635-1646.
Scheidegger, A.E. 1961. Mathematical models of slope development, *Bulletin of the Geological Society of America,* **72**, 37-59.
Scheidegger, A.E. 1992. Limitations of the system approach in geomorphology, *Geomorphology,* **5**, 213-217.
Schumm, S.A., Mosley, M.P. and Weaver, W.E., 1987. *Experimental Fluvial Geomorphology,* Wiley, New York, 413 pp.
Smith, T.R. and Bretherton, F.P. 1972. Stability and the conservation of mass in drainage basin evolution, *Water Resources Research,* **8**, 1506-1529.
Strahler, A.N. 1952. Hypsometric (area-altitude) analysis of erosional topography, *Bulletin of the Geological Society of America,* **63**, 1117-1142.

Tarboton, D.G., Bras, R.L. and Rodriguez-Iturbe, I. 1988. The fractal nature of river networks, *Water Resources Research,* **24***,* 1317-1322.

Thomas, P.W. and Huggett, R.J. 1980. *Modelling in Geography: A Mathematical Approach,* Harper and Row, London, 338 pp,

Willgoose, G., Bras, R.L. and Rodriguez-Iturbe, I. 1990. A model of river basin evolution. EOS, *Transactions of the American Geophysical Union,* **71**, 1806-1807.

Willgoose, G., Bras, R.L. and Rodriguez-Iturbe, I. 1991a. A coupled channel network growth and hillslope evolution model. 1. Theory, *Water Resources Research,* **27**, 1671-1684.

Willgoose, G., Bras, R.L. and Rodriguez-Iturbe, I. 1991b. A coupled channel network growth and hillslope evolution model. 2. Nondimensionalization and applications, *Water Resources Research,* **27**, 1685-1696.